

UNIVERSALLY CATENARIAN INTEGRAL DOMAINS, STRONG S-DOMAINS AND SEMISTAR OPERATIONS

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ABSTRACT. Let D be an integral domain and \star a semistar operation stable and of finite type on it. In this paper, we are concerned with the study of the semistar (Krull) dimension theory of polynomial rings over D . We introduce and investigate the notions of \star -universally catenarian and \star -stably strong S-domains and prove that, every \star -locally finite dimensional Prüfer \star -multiplication domain is \star -universally catenarian, and this implies \star -stably strong S-domain. We also give new characterizations of \star -quasi-Prüfer domains introduced recently by Chang and Fontana, in terms of these notions.

1. INTRODUCTION

The concepts of S(eidenberg)-domains and strong S-domains are crucial ones and were introduced by Kaplansky [18, Page 26]. Recall that an integral domain D is an *S-domain* if for each prime ideal P of D of height one the extension $PD[X]$ to the polynomial ring in one variable is also of height one. A *strong S-domain* is a domain D such that, D/P is an S-domain, for each prime P of D . One of the reasons why Kaplansky introduced the notion of strong S-domain was to treat the classes of Noetherian domains and Prüfer domains in a unified frame. Moreover, if D belongs to one of the two classes of domains, then the following dimension formula holds: $\dim(D[X_1, \dots, X_n]) = n + \dim(D)$ (cf., [23, Theorem 9] and [24, Theorem 4]). The integral domain D is called a *Jaffard domain* if $\dim(D) < \infty$ and $\dim(D[X_1, \dots, X_n]) = n + \dim(D)$ for each positive integer n . So that finite dimensional Noetherian or Prüfer domains are Jaffard domains. Kaplansky observed that for $n = 1$ and for D a strong S-domain then $\dim(D[X_1]) = 1 + \dim(D)$ [18, Theorem 39]. The strong S-property is not stable, in general under polynomial extensions (cf. [6]). In [19], Malik and Mott, defined and studied the stably strong S-domains. A domain D is called a *stably strong S-domain* if $D[X_1, \dots, X_n]$ is a strong S-domain for each $n \geq 1$. Note that the class of Jaffard domains contains the class of stably strong S-domains. The class of stably strong S-domains contains an important class of universally catenarian domains. Recall that a domain D , is called *catenarian*, if for each pair $P \subset Q$ of prime ideals of D , any two saturated chain of prime ideals between P and Q have the same finite length. If for each $n \geq 1$, the polynomial ring $D[X_1, \dots, X_n]$ is catenary, then D is said to be *universally catenarian* (cf. [4, 3]).

For several decades, star operations, as described in [17, Section 32], have proven to be an essential tool in *multiplicative ideal theory*, for studying various classes of

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domains. In [20], Okabe and Matsuda introduced the concept of a semistar operation to extend the notion of a star operation. Since then, semistar operations have been extensively studied and, because of a greater flexibility than star operations, have permitted a finer study and new classifications of special classes of integral domains.

This manuscript is a sequel to [22]. Given a semistar operation \star on D and let $\tilde{\star}$ be the stable semistar operation of finite type canonically associated to \star (the definitions are recalled later in this section), it is possible to define a semistar operation stable and of finite type $\star[X]$ on $D[X]$ (cf. [22]) such that:

$$\tilde{\star}\text{-dim}(D) + 1 \leq \star[X]\text{-dim}(D[X]) \leq 2(\tilde{\star}\text{-dim}(D)) + 1.$$

We say that a domain D , is an $\tilde{\star}$ -Jaffard domain if $\tilde{\star}\text{-dim}(D) < \infty$ and

$$\star[X_1, \dots, X_n]\text{-dim}(D[X_1, \dots, X_n]) = \tilde{\star}\text{-dim}(D) + n,$$

for each positive integer n . Every $\tilde{\star}$ -Noetherian and P \star MDs are $\tilde{\star}$ -Jaffard domains (cf. [22]). In this paper we define and study two subclass of $\tilde{\star}$ -Jaffard domains. Namely in Sections 2 and 3, we define and study $\tilde{\star}$ -stably strong S-domains and $\tilde{\star}$ -universally catenarian domains. In Section 4 we give new characterizations of $\tilde{\star}$ -quasi-Prüfer domains in terms of $\tilde{\star}$ -stably strong S-domains and $\tilde{\star}$ -universally catenarian domains.

To facilitate the reading of the introduction and of the paper, we first review some basic facts on semistar operations. Let D denote a (commutative integral) domain with identity and let K be the quotient field of D . Denote by $\overline{\mathcal{F}}(D)$ the set of all nonzero D -submodules of K , and by $\mathcal{F}(D)$ the set of all nonzero *fractional ideals* of D ; i.e., $E \in \mathcal{F}(D)$ if $E \in \overline{\mathcal{F}}(D)$ and there exists a nonzero element $r \in D$ with $rE \subseteq D$. Let $f(D)$ be the set of all nonzero finitely generated fractional ideals of D . Obviously, $f(D) \subseteq \mathcal{F}(D) \subseteq \overline{\mathcal{F}}(D)$. As in [20], a *semistar operation* on D is a map $\star : \overline{\mathcal{F}}(D) \rightarrow \overline{\mathcal{F}}(D)$, $E \mapsto E^\star$, such that, for all $x \in K$, $x \neq 0$, and for all $E, F \in \overline{\mathcal{F}}(D)$, the following three properties hold:

- $\star_1 : (xE)^\star = xE^\star$;
- $\star_2 : E \subseteq F$ implies that $E^\star \subseteq F^\star$;
- $\star_3 : E \subseteq E^\star$ and $E^{\star\star} := (E^\star)^\star = E^\star$.

It is convenient to say that a *(semi)star operation* on D is a semistar operation which, when restricted to $\mathcal{F}(D)$, is a star operation (in the sense of [17, Section 32]). It is easy to see that a semistar operation \star on D is a (semi)star operation on D if and only if $D^\star = D$.

Let \star be a semistar operation on the domain D . For every $E \in \overline{\mathcal{F}}(D)$, put $E^{\star_f} := \bigcup F^\star$, where the union is taken over all finitely generated $F \in f(D)$ with $F \subseteq E$. It is easy to see that \star_f is a semistar operation on D , and \star_f is called *the semistar operation of finite type associated to \star* . Note that $(\star_f)_f = \star_f$. A semistar operation \star is said to be of *finite type* if $\star = \star_f$; in particular \star_f is of finite type. We say that a nonzero ideal I of D is a *quasi- \star -ideal* of D , if $I^\star \cap D = I$; a *quasi- \star -prime* (ideal of D), if I is a prime quasi- \star -ideal of D ; and a *quasi- \star -maximal* (ideal of D), if I is maximal in the set of all proper quasi- \star -ideals of D . Each quasi- \star -maximal ideal is a prime ideal. It was shown in [12, Lemma 4.20] that if $D^\star \neq K$, then each proper quasi- \star_f -ideal of D is contained in a quasi- \star_f -maximal ideal of D . We denote by $\text{QMax}^\star(D)$ (resp., $\text{QSpec}^\star(D)$) the set of all quasi- \star -maximal ideals (resp., quasi- \star -prime ideals) of D . When \star is a (semi)star operation, it is easy to

see that the notion of quasi- \star -ideal is equivalent to the classical notion of \star -ideal (i.e., a nonzero ideal I of D such that $I^\star = I$).

If Δ is a set of prime ideals of a domain D , then there is an associated semistar operation on D , denoted by \star_Δ , defined as follows:

$$E^{\star_\Delta} := \cap\{ED_P \mid P \in \Delta\}, \text{ for each } E \in \overline{\mathcal{F}}(D).$$

If $\Delta = \emptyset$, let $E^{\star_\Delta} := K$ for each $E \in \overline{\mathcal{F}}(D)$. One calls \star_Δ the *spectral semistar operation associated to Δ* . A semistar operation \star on a domain D is called a *spectral semistar operation* if there exists a subset Δ of the prime spectrum of D , $\text{Spec}(D)$, such that $\star = \star_\Delta$. When $\Delta := \text{QMax}^{\star_f}(D)$, we set $\tilde{\star} := \star_\Delta$; i.e.,

$$E^{\tilde{\star}} := \cap\{ED_P \mid P \in \text{QMax}^{\star_f}(D)\}, \text{ for each } E \in \overline{\mathcal{F}}(D).$$

It has become standard to say that a semistar operation \star is *stable* if $(E \cap F)^\star = E^\star \cap F^\star$ for all $E, F \in \overline{\mathcal{F}}(D)$. All spectral semistar operations are stable [12, Lemma 4.1(3)]. In particular, for any semistar operation \star , we have that $\tilde{\star}$ is a stable semistar operation of finite type [12, Corollary 3.9].

The most widely studied (semi)star operations on D have been the identity d_D , and v_D , $t_D := (v_D)_f$, and $w_D := \widetilde{v_D}$ operations, where $E^{v_D} := (E^{-1})^{-1}$, with $E^{-1} := (D : E) := \{x \in K \mid xE \subseteq D\}$.

Let \star be a semistar operation on a domain D . The \star -Krull dimension of D is defined as

$$\star\text{-dim}(D) := \sup \left\{ n \mid \begin{array}{l} (0) = P_0 \subset P_1 \subset \cdots \subset P_n \text{ where } P_i \text{ is a} \\ \text{quasi-}\star\text{-prime ideal of } D \text{ for } 1 \leq i \leq n \end{array} \right\}.$$

It is known (see [11, Lemma 2.11]) that

$$\tilde{\star}\text{-dim}(D) = \sup\{\text{ht}(P) \mid P \text{ is a quasi-}\tilde{\star}\text{-prime ideal of } D\}.$$

Thus, if $\star = d_D$, then $\tilde{\star}\text{-dim}(D) = \star\text{-dim}(D)$ coincides with $\text{dim}(D)$, the usual (Krull) dimension of D .

Let \star be a semistar operation on a domain D . Recall from [11, Section 3] that D is said to be a \star -Noetherian domain, if D satisfies the ascending chain condition on quasi- \star -ideals. Also recall from [14] that, D is called a *Prüfer \star -multiplication domain* (for short, a $\text{P}\star\text{MD}$) if each finitely generated ideal of D is \star_f -invertible; i.e., if $(II^{-1})^{\star_f} = D^\star$ for all $I \in f(D)$. When $\star = v$, we recover the classical notion of PvMD ; when $\star = d_D$, the identity (semi)star operation, we recover the notion of Prüfer domain.

Let D be a domain, \star a semistar operation on D , T an overring of D , and $\iota : D \hookrightarrow T$ the corresponding inclusion map. In a canonical way, one can define an associated semistar operation \star_ι on T , by setting $E \mapsto E^{\star_\iota} := E^\star$, for each $E \in \overline{\mathcal{F}}(T) (\subseteq \overline{\mathcal{F}}(D))$ [16, Proposition 2.8].

Throughout this paper, D denotes a domain and \star is a semistar operation on D .

2. THE \star -STRONG S-DOMAINS

Let D be an integral domain with quotient field K , let X, Y be two indeterminates over D and let \star be a semistar operation on D . Set $D_1 := D[X]$, $K_1 := K(X)$ and take the following subset of $\text{Spec}(D_1)$:

$$\Theta_1^\star := \{Q_1 \in \text{Spec}(D_1) \mid Q_1 \cap D = (0) \text{ or } (Q_1 \cap D)^{\star_f} \subsetneq D^\star\}.$$

Set $\mathfrak{S}_1^* := \mathcal{S}(\Theta_1^*) := D_1[Y] \setminus (\bigcup \{Q_1[Y] \mid Q_1 \in \Theta_1^*\})$ and:

$$E^{\circ \mathfrak{S}_1^*} := E[Y]_{\mathfrak{S}_1^*} \cap K_1, \text{ for all } E \in \overline{\mathcal{F}}(D_1).$$

It is proved in [22, Theorem 2.1] that the mapping $\star[X] := \circ \mathfrak{S}_1^* : \overline{\mathcal{F}}(D_1) \rightarrow \overline{\mathcal{F}}(D_1)$, $E \mapsto E^{\star[X]}$ is a stable semistar operation of finite type on $D[X]$, i.e., $\star[X] = \widetilde{\star[X]}$. It is also proved that $\tilde{\star}[X] = \star_f[X] = \star[X]$, $d_D[X] = d_{D[X]}$ and $\text{QSpec}^{\star[X]}(D[X]) = \Theta_1^* \setminus \{0\}$. If X_1, \dots, X_r are indeterminates over D , for $r \geq 2$, we let

$$\star[X_1, \dots, X_r] := (\star[X_1, \dots, X_{r-1}])[X_r],$$

where $\star[X_1, \dots, X_{r-1}]$ is a stable semistar operation of finite type on $D[X_1, \dots, X_{r-1}]$. For an integer r , put $\star[r]$ to denote $\star[X_1, \dots, X_r]$ and $D[r]$ to denote $D[X_1, \dots, X_r]$.

As an extension of a result by Seidenberg [23, Theorem 2], we showed in [22, Theorem 3.1] that: if $n := \tilde{\star}\text{-dim}(D)$, then $n + 1 \leq \star[X]\text{-dim}(D[X]) \leq 2n + 1$. On the other hand, it is shown in [22, Theorem 3.8 and Corollary 4.11], that if D is a $\tilde{\star}$ -Noetherian domain or a $\text{P}\star\text{MD}$ and n is any positive integer, then $\star[n]\text{-dim}(D[n]) = n + \tilde{\star}\text{-dim}(D)$, that is D is an $\tilde{\star}$ -Jaffard domain. Now we define and study a subclass of $\tilde{\star}$ -Jaffard domains.

Definition 2.1. *The domain D is called an \star -S-domain, if each height one quasi- \star -prime ideal P of D , extends to a height one quasi- $\star[X]$ -prime ideal $P[X]$ of the polynomial ring $D[X]$. We say that D is an \star -strong S-domain, if each pair of adjacent quasi- \star -prime ideals $P_1 \subset P_2$ of D , extend to a pair of adjacent quasi- $\star[X]$ -prime ideals $P_1[X] \subset P_2[X]$, of $D[X]$. If for each $n \geq 1$, the polynomial ring $D[n]$ is a $\star[n]$ -strong S-domain, then D is said to be an \star -stably strong S-domain.*

Note that the notion of d -S-domain (resp. d -strong S-domain, d -stably strong S-domain) coincides with the “classical” notion of S-domain (resp. strong S-domain, stably strong S-domain) [18, 19].

Proposition 2.2. *Suppose that D is an $\tilde{\star}$ -strong S-domain, and $\tilde{\star}\text{-dim}(D) = n$ is finite. Then $\star[X]\text{-dim}(D[X]) = n + 1$.*

Proof. By [22, Theorem 3.1], we only have to show that $\star[X]\text{-dim}(D[X]) \leq n + 1$. Let $Q \in \text{QSpec}^{\star[X]}(D[X])$ and set $P := Q \cap D$. There are two cases to consider. If $P = 0$, then by [18, Theorem 37], $\text{ht}(Q) \leq 1$. If $P \neq 0$, we show that $P \in \text{QSpec}^{\tilde{\star}}(D)$. Let $M \in \text{QMax}^{\star[X]}(D[X])$ containing Q [15, Lemma 2.3 (1)]. Since $0 \neq P \subseteq M \cap D$, then $M \cap D \in \text{QSpec}^{\star_f}(D)$ by [22, Remark 2.3]. Hence $P \in \text{QSpec}^{\tilde{\star}}(D)$ by [12, Lemma 4.1, Remark 4.5]. Consequently $\text{ht}(P) \leq n$ by the hypothesis. By an argument the same as [18, Theorem 39], we obtain that $\text{ht}(P) = \text{ht}(P[X])$. Since $P[X] \subseteq Q$, then we have $\text{ht}(Q) \leq n + 1$. Therefore $\star[X]\text{-dim}(D[X]) \leq n + 1$ as desired. \square

Corollary 2.3. *Each $\tilde{\star}$ -stably strong S-domain of finite $\tilde{\star}$ -dimension is an $\tilde{\star}$ -Jaffard domain.*

Proposition 2.4. *Let D be an integral domain. The following then are equivalent:*

- (1) D is an $\tilde{\star}$ -S-domain (resp. $\tilde{\star}$ -strong S-domain).
- (2) D_P is an S-domain (resp. strong S-domain) for all $P \in \text{QSpec}^{\tilde{\star}}(D)$.
- (3) D_M is an S-domain (resp. strong S-domain) for all $M \in \text{QMax}^{\tilde{\star}}(D)$.

Proof. For either cases follow the method of [19, Proposition 2.1], and note that every quasi- $\tilde{\star}$ -prime ideal of D is contained in a quasi- $\tilde{\star}$ -maximal ideal by [15, Lemma 2.3 (1)]. \square

Proposition 2.5. *Let D be an integral domain. The following then are equivalent:*

- (1) D is an $\tilde{\star}$ -stably strong S-domain.
- (2) D_P is an stably strong S-domain, for all $P \in \text{QSpec}^{\tilde{\star}}(D)$.
- (3) D_M is an stably strong S-domain, for all $M \in \text{QMax}^{\tilde{\star}}(D)$.

Proof. (1) \Rightarrow (2). Suppose that D is an $\tilde{\star}$ -stably strong S-domain, $P \in \text{QSpec}^{\tilde{\star}}(D)$ and $n \geq 1$ is an integer. It suffices by [13, Lemma 6.3.1], to show that for each maximal ideal \mathcal{M} of $D_P[n]$, the local ring $D_P[n]_{\mathcal{M}}$ is a strong S-domain. To this end, let \mathcal{M} be an arbitrary maximal ideal of $D_P[n]$. Note that $D_P[n] = D[n]_{D \setminus P}$. So that there exists a prime ideal M of $D[n]$ such that $M \cap (D \setminus P) = \emptyset$ and $\mathcal{M} = MD_P[n]$. Consequently $M \cap D \subseteq P$, and therefore by [22, Remark 2.3], we have $M \in \text{QSpec}^{\star[n]}(D[n])$. Now since by the hypothesis $D[n]$ is an $\star[n]$ -strong S-domain, we then have $D_P[n]_{\mathcal{M}} = D[n]_M$ is a strong S-domain by Proposition 2.4.

(2) \Rightarrow (1). Let $n \geq 1$ be an integer, $Q \in \text{QSpec}^{\star[n]}(D[n])$ and set $P := Q \cap D$. We plan to show that $D[n]_Q$ is an strong S-domain. If $P = 0$, then $D[n]_Q = K[n]_{QK[n]}$, which is an strong S-domain since it is Noetherian and [18, Theorem 149]. If $P \neq 0$, then $P \in \text{QSpec}^{\tilde{\star}}(D)$. Therefore $D[n]_Q = D_P[n]_{QD_P[n]}$, and hence is an strong S-domain by hypothesis. So that $D[n]$ is an $\star[n]$ -strong S-domain by Proposition 2.4. Thus D is an $\tilde{\star}$ -stably strong S-domain by the definition.

(2) \Leftrightarrow (3) is true, using [13, Lemma 6.3.1]. \square

Recall from [7] that D is said to be a \star -quasi-Prüfer domain, in case, if Q is a prime ideal in $D[X]$, and $Q \subseteq P[X]$, for some $P \in \text{QSpec}^{\star}(D)$, then $Q = (Q \cap D)[X]$. This notion is the semistar analogue of the classical notion of the quasi-Prüfer domains [13, Section 6.5] (that is among other equivalent conditions, the domain D is said to be a quasi-Prüfer domain if it has Prüferian integral closure). By [7, Corollary 2.4], D is a \star_f -quasi-Prüfer domain if and only if D is a $\tilde{\star}$ -quasi-Prüfer domain.

Corollary 2.6. *If D is a $\tilde{\star}$ -Noetherian (resp. a $\tilde{\star}$ -quasi-Prüfer) domain, then D is an $\tilde{\star}$ -stably strong S-domain.*

Proof. Let $P \in \text{QSpec}^{\tilde{\star}}(D)$. Since D_P is a Noetherian domain by [11, Proposition 3.8] (resp. a quasi-Prüfer domain by [7, Lemma 2.1]), we obtain that it is an stably strong S-domain by [18, Theorem 149] (resp. by [13, Corollary 6.7.6]). Therefore D is an $\tilde{\star}$ -stably strong S-domain by Proposition 2.5. \square

Therefore we have the following implications for finite $\tilde{\star}$ -dimensional domains:

$$\tilde{\star}\text{-Noetherian or } \tilde{\star}\text{-quasi-Prüfer} \Rightarrow \tilde{\star}\text{-stably strong S-domain} \Rightarrow \tilde{\star}\text{-Jaffard}.$$

Remark 2.7. *Call a domain D , an $\tilde{\star}$ -locally Jaffard domain if D_P is a Jaffard domain for each $P \in \text{QSpec}^{\tilde{\star}}(D)$. Therefore every $\tilde{\star}$ -stably strong S-domain is a $\tilde{\star}$ -locally Jaffard domain. It is not hard to prove that every $\tilde{\star}$ -locally Jaffard domain is an $\tilde{\star}$ -Jaffard domain (see proof of [22, Theorem 3.2]).*

3. THE \star -CATENARIAN DOMAINS

In this section we introduce and study a subclass of $\tilde{\star}$ -stably strong S-domains, namely $\tilde{\star}$ -universally catenarian domains.

Definition 3.1. *The domain D is called \star -catenary, if for each pair $P \subset Q$ of quasi- \star -prime ideals of D , any two saturated chain of quasi- \star -prime ideals between P and Q have the same finite length. If for each $n \geq 1$, the polynomial ring $D[n]$ is $\star[n]$ -catenary, then D is said to be \star -universally catenarian.*

Note that the notion of d -catenary (resp. d -universally catenarian) coincides with the “classical” notion of catenary (resp. universally catenarian). The proof of the the following proposition is straightforward, so we omit it.

Proposition 3.2. *Let D be an integral domain. The following then are equivalent:*

- (1) D is $\tilde{\star}$ -catenary.
- (2) D_P is catenary for all $P \in \text{QSpec}^{\tilde{\star}}(D)$.
- (3) D_M is catenary for all $M \in \text{QMax}^{\tilde{\star}}(D)$.

Lemma 3.3. *Let D be an integral domain and $n \geq 1$ be an integer. Then $D[n]$ is $\star[n]$ -catenary, if and only if, $D_P[n]$ is catenary for all $P \in \text{QSpec}^{\tilde{\star}}(D)$.*

Proof. (\Rightarrow) It suffices by [13, Lemma 6.3.2], to show that for each maximal ideal \mathcal{M} of $D_P[n]$, the local ring $D_P[n]_{\mathcal{M}}$ is catenary. To this end, let \mathcal{M} be an arbitrary maximal ideal of $D_P[n]$. Note that $D_P[n] = D[n]_{D \setminus P}$. So that there exists a prime ideal M of $D[n]$ such that $M \cap (D \setminus P) = \emptyset$ and $\mathcal{M} = MD_P[n]$. Consequently $M \cap D \subseteq P$, and therefore by [22, Remark 2.3], we have $M \in \text{QSpec}^{\star[n]}(D[n])$. Hence $D_P[n]_{\mathcal{M}} = D[n]_M$ is a catenary domain.

(\Leftarrow) Let $Q \in \text{Max}^{\star[n]}(D[n])$ and set $P := Q \cap D$. We plan to show that $D[n]_Q$ is a catenarian domain. There are two cases to consider. If $P = 0$, then $D[n]_Q = K[n]_{QK[n]}$, which is a catenarian domain since it is a Cohen-Macaulay ring and [5, Theorem 2.1.12]. If $P \neq 0$, then $P \in \text{QSpec}^{\tilde{\star}}(D)$. Therefore $D[n]_Q = D_P[n]_{QD_P[n]}$, which is catenary by the hypothesis. Whence $D[n]$ is $\star[n]$ -catenary by Proposition 3.2. \square

It is convenient to say that, a domain D is $\tilde{\star}$ -locally finite dimensional (for short, $\tilde{\star}$ -LFD) if $\text{ht}(P) < \infty$ for every $P \in \text{QSpec}^{\tilde{\star}}(D)$. The special case $\star = d_D$ of the following theorem is contained in [3, Theorem 12].

Theorem 3.4. *If D is a $P\star MD$ which is $\tilde{\star}$ -LFD, then D is $\tilde{\star}$ -universally catenarian.*

Proof. We have to show that for each integer $n \geq 1$, $D[n]$ is a $\star[n]$ -catenarian domain. To this end let $n \geq 1$ be an integer and $P \in \text{QSpec}^{\tilde{\star}}(D)$. So that D_P is a finite dimensional valuation domain by the hypothesis and [14, Theorem 3.1]. Hence $D_P[n]$ is catenary by [3, Theorem 12]. Thus $D[n]$ is $\star[n]$ -catenary by Lemma 3.3, for all $n \geq 1$. Hence D is $\tilde{\star}$ -universally catenarian. \square

Proposition 3.5. *Let D be an integral domain. If $D[X]$ is $\star[X]$ -catenarian, then D is an $\tilde{\star}$ -strong S-domain.*

Proof. Using Lemma 3.3, $D_P[X]$ is a catenarian domain for all $P \in \text{QSpec}^{\tilde{\star}}(D)$. Hence D_P is a strong S-domain by [4, Lemma 2.3]. Thus D is a $\tilde{\star}$ -strong S-domain by Proposition 2.4. \square

Corollary 3.6. *Each $\tilde{\star}$ -universally catenarian domain is an $\tilde{\star}$ -stably strong S-domain.*

Therefore we have the following implications for finite $\tilde{\star}$ -dimensional domains:

$$P\star MD \Rightarrow \tilde{\star}\text{-universally catenarian} \Rightarrow \tilde{\star}\text{-stably strong S-domain} \Rightarrow \tilde{\star}\text{-Jaffard}.$$

Next we wish to present the semistar analogue of the celebrated theorem of Ratliff [21, Theorem 2.6].

Theorem 3.7. *Let D be an integral domain. Suppose that D is $\tilde{\star}$ -Noetherian. Then $D[X]$ is $\star[X]$ -catenarian if and only if D is $\tilde{\star}$ -universally catenarian.*

Proof. The “if” part is trivial. For the “only if” part, let $n \geq 1$ be an integer and $P \in \text{QSpec}^{\tilde{\star}}(D)$. So by Lemma 3.3, $D_P[X]$ is catenarian. Since D_P is a Noetherian domain by [11, Proposition 3.8], using the result of Ratliff [21, Theorem 2.6], we have D_P is a universally catenarian domain. Therefore $D_P[n]$ is a catenarian domain. Thus another use of Lemma 3.3 yields us that $D[n]$ is a $\star[n]$ -catenarian domain for all $n \geq 1$. Hence D is an $\tilde{\star}$ -universally catenarian. \square

4. CHARACTERIZATIONS OF \star -QUASI-PRÜFER DOMAINS

In this section we give some characterization of $\tilde{\star}$ -quasi-Prüfer domains. First we need to recall the definition of a semistar going-down domain. Let $D \subseteq T$ be an extension of domains. Let \star and \star' be semistar operations on D and T , respectively. Following [9], we say that $D \subseteq T$ satisfies (\star, \star') -GD if, whenever $P_0 \subset P$ are quasi- \star -prime ideals of D and Q is a quasi- \star' -prime ideal of T such that $Q \cap D = P$, there exists a quasi- \star -prime ideal Q_0 of T such that $Q_0 \subseteq Q$ and $Q_0 \cap D = P_0$. The integral domain D is said to be a \star -going-down domain (for short, a \star -GD domain) if, for every overring T of D and every semistar operation \star' on T , the extension $D \subseteq T$ satisfies (\star, \star') -GD. These concepts are the semistar versions of the “classical” concepts of going-down property and the going-down domains (cf. [8]). It is known by [9, Propositions 3.5 and 3.2(e)] that every $P\star MD$ and every integral domain D with $\star\text{-dim}(D) = 1$ is a \star -GD domain.

Theorem 4.1. *Let D be an integral domain. Suppose that D is a $\tilde{\star}$ -GD domain which is $\tilde{\star}$ -LFD. Then the following statements are equivalent:*

- (1) D is an $\tilde{\star}$ -universally catenarian domain.
- (2) D is an $\tilde{\star}$ -stably strong S-domain.
- (3) D is an $\tilde{\star}$ -strong S-domain.
- (4) D is an $\tilde{\star}$ -locally Jaffard domain.
- (5) D_M is a Jaffard domain for each $M \in \text{QMax}^{\tilde{\star}}(D)$.
- (6) $D[X]$ is an $\star[X]$ -catenarian domain.
- (7) D is an $\tilde{\star}$ -quasi-Prüfer domain.

Proof. First of all we show that for each $P \in \text{QSpec}^{\tilde{\star}}(D)$, D_P is a going-down domain. Let T be an overring of D_P . Suppose that $P_1 D_P \subset P_2 D_P$ are prime ideals of D_P and Q_2 is a prime ideal of T such that $Q_2 \cap D_P = P_2 D_P$. Since $P_1 \subset P_2$ are quasi- $\tilde{\star}$ -prime ideals of D (since they are contained in P and [12, Lemma 4.1 and Remark 4.5]) and $Q_2 \cap D = P_2$ and the fact that D is a $\tilde{\star}$ -GD domain, there exists a (quasi- $\tilde{\star}$ -)prime ideal Q_1 of T satisfying both $Q_1 \subseteq Q_2$ and $Q_1 \cap D = P_1$. So that $Q_1 \cap D_P = P_1 D_P$. Therefore D_P is a going-down domain.

The implications $(1) \Rightarrow (2)$, $(2) \Rightarrow (3)$, and $(4) \Rightarrow (5)$ are already known (see Section 3).

$(3) \Rightarrow (4)$. Let $P \in \text{QSpec}^{\tilde{\star}}(D)$. Therefore D_P is a going-down domain which is also a strong S-domain by Proposition 2.4. Hence by [1, Theorem 1.13], D_P is a Jaffard domain. Thus D is an $\tilde{\star}$ -locally Jaffard.

$(5) \Rightarrow (6)$. Let $P \in \text{QSpec}^{\tilde{\star}}(D)$. Choose a quasi- $\tilde{\star}$ -maximal ideal M of D containing P . Since D_M is a Jaffard domain which is also going-down, then [1, Theorem 1.13] tells us that $D_M[X]$ is catenarian. Thus $D_P[X] = (D_M[X])_{D \setminus P}$ is a catenarian domain. Now by Lemma 3.3, $D[X]$ is $\star[X]$ -catenarian.

$(6) \Rightarrow (7)$. Let $P \in \text{QSpec}^{\tilde{\star}}(D)$. Using Lemma 3.3, $D_P[X]$ is catenarian. Since D_P is a going-down domain, using [1, Theorem 1.13], we see that D_P is a quasi-Prüfer domain. Thus by [7, Lemma 2.1], D is a $\tilde{\star}$ -quasi-Prüfer domain.

$(7) \Rightarrow (1)$. Let $P \in \text{QSpec}^{\tilde{\star}}(D)$. Since D_P is a quasi-Prüfer going-down domain, we have D_P is a universally catenarian domain by [1, Theorem 1.13]. Hence $D_P[n]$ is a catenarian domain for each integer $n \geq 1$. Therefore using Lemma 3.3 we obtain that $D[n]$ is a $\star[n]$ -catenarian domain for all $n \geq 1$. Hence D is an $\tilde{\star}$ -universally catenarian domain. \square

Corollary 4.2. *Let D be an integral domain. Suppose that $\tilde{\star}\text{-dim}(D) = 1$. Then the following statements are equivalent:*

- (1) D is an $\tilde{\star}$ -universally catenarian domain.
- (2) $D[X]$ is an $\star[X]$ -catenarian domain.
- (3) D is an $\tilde{\star}$ -stably strong S-domain.
- (4) D is an $\tilde{\star}$ -strong S-domain.
- (5) D is an $\tilde{\star}$ -S-domain.
- (6) $\star[X]\text{-dim}(D[X]) = 2$.
- (7) D is an $\tilde{\star}$ -Jaffard domain.
- (8) D is an $\tilde{\star}$ -locally Jaffard domain.
- (9) D is an $\tilde{\star}$ -quasi-Prüfer domain.

Proof. Observe that $(1) \Leftrightarrow (2) \Leftrightarrow (3) \Leftrightarrow (4) \Leftrightarrow (8) \Leftrightarrow (9)$ by Theorem 4.1, since D is a $\tilde{\star}$ -GD domain and $(4) \Rightarrow (5)$ is trivial.

$(5) \Rightarrow (6)$. By [22, Theorem 3.1], we have $2 \leq \star[X]\text{-dim}(D[X])$. Now let P be a quasi- $\tilde{\star}$ -prime ideal of D . Hence by the hypothesis, $\text{ht}(P) = 1$. Since D is a $\tilde{\star}$ -S-domain, $\text{ht}(P[X]) = 1$. Now let Q be a quasi- $\star[X]$ -prime ideal of $D[X]$. Take a prime ideal Q_1 properly contained in Q . If it contracts to a $P \in \text{QSpec}^{\tilde{\star}}(D)$, then by [18, Theorem 37], $Q_1 = P[X]$. If it contracts to zero, then by [18, Theorem 37], we have $\text{ht}(Q_1) \leq 1$. Whence in either cases we have $\text{ht}(Q) \leq 2$. Therefore $\star[X]\text{-dim}(D[X]) \leq 2$. Thus $\star[X]\text{-dim}(D[X]) = 2$.

$(6) \Leftrightarrow (7)$ is true by [22, Corollary 4.12] and $(6) \Rightarrow (9)$ by [22, Theorem 3.5]. \square

Next we characterize $\tilde{\star}$ -quasi-Prüfer domains by the means of the properties of their overrings. Before that we need to recall the definition of (\star, \star') -linked overrings. Let D be a domain and T an overring of D . Let \star and \star' be semistar operations on D and T , respectively. One says that T is (\star, \star') -linked to D (or that T is a (\star, \star') -linked overring of D) if $F^\star = D^\star \Rightarrow (FT)^{\star'} = T^{\star'}$, when F is a nonzero finitely generated ideal of D (cf. [10]).

Let T be a (\star, \star') -linked overring of D . We will show that the contraction map on prime spectra restricts to a well defined function

$$G : \text{QSpec}^{\tilde{\star}'}(T) \rightarrow \text{QSpec}^{\tilde{\star}}(D), \quad Q \mapsto Q \cap D$$

of topological spaces which is continuous (with respect to the subspace topology induced by the Zariski topology). If $Q \in \text{QSpec}^{\tilde{\star}'}(T)$, then we show that $P := G(Q) = Q \cap D$ is a quasi- $\tilde{\star}$ -ideal of D . To this end it suffices to show that $P^{\tilde{\star}} \neq D^{\tilde{\star}}$. So suppose that $P^{\tilde{\star}} = D^{\tilde{\star}}$. Then $(PT)^{\tilde{\star}'} = T^{\tilde{\star}'}$. Since $PT \subseteq Q$ we obtain that $Q^{\tilde{\star}'} = T^{\tilde{\star}'}$, and hence $Q = T$, which is a contradiction.

In the following theorem, let \star' be a semistar operation for an overring T of D .

Theorem 4.3. *Let D be an integral domain. Suppose that $\tilde{\star}\text{-dim}(D)$ is finite. Then the following statements are equivalent:*

- (1) *Each (\star, \star') -linked overring T of D is an $\tilde{\star}'$ -universally catenarian domain.*
- (2) *Each (\star, \star') -linked overring T of D is an $\tilde{\star}'$ -stably strong S-domain.*
- (3) *Each (\star, \star') -linked overring T of D is an $\tilde{\star}'$ -strong S-domain.*
- (4) *Each (\star, \star') -linked overring T of D is an $\tilde{\star}'$ -Jaffard domain.*
- (5) *Each (\star, \star') -linked overring T of D is an $\tilde{\star}'$ -quasi-Prüfer domain.*
- (6) *D is an $\tilde{\star}$ -quasi-Prüfer domain.*

Proof. Note that (1) \Rightarrow (2) and (2) \Rightarrow (3) are trivial.

(3) \Rightarrow (6). Let M be a quasi- \star_f -maximal ideal of D . We wish to show that D_M is a quasi-Prüfer domain. Suppose that T is an overring of D_M . Since T is a (\star, \star_ι) -linked overring of D , we have T is a $\tilde{\star}_\iota$ -strong S-domain by the hypothesis, where ι is the canonical inclusion of D into T . We want to show that $\text{QSpec}^{\tilde{\star}_\iota}(T) \cup \{0\} = \text{Spec}(T)$. So let Q be an arbitrary non-zero prime ideal of T , and set $PD_M := Q \cap D_M$, where $P \in \text{Spec}(D)$ such that $P \subseteq M$. Note that P is a quasi- $\tilde{\star}$ -prime ideal of D , since it is contained in M and [12, Lemma 4.1 and Remark 4.5], and that $P = Q \cap D$. If $Q^{\tilde{\star}_\iota} = T^{\tilde{\star}_\iota}$, that is, if $Q^{\tilde{\star}} = T^{\tilde{\star}}$, then we have $Q^{\tilde{\star}} \cap D = D$. But

$$P = P^{\tilde{\star}} \cap D = (Q \cap D)^{\tilde{\star}} \cap D = Q^{\tilde{\star}} \cap D = D,$$

which is a contradiction. Therefore $Q^{\tilde{\star}_\iota} \neq T^{\tilde{\star}_\iota}$, and hence $Q \in \text{QSpec}^{\tilde{\star}_\iota}(T)$ since $\tilde{\star}_\iota = \widetilde{(\star_\iota)}$ is a stable semistar operation of finite type, and so $\text{QSpec}^{\tilde{\star}_\iota}(T) \cup \{0\} = \text{Spec}(T)$. This means that T is a strong S-domain. Therefore thanks to [13, Theorem 6.7.8], D_M is a quasi-Prüfer domain. Hence D is a $\tilde{\star}$ -quasi-Prüfer domain by [7, Lemma 2.1].

(4) \Leftrightarrow (5) \Leftrightarrow (6) was proved in [22, Theorem 4.14].

(6) \Rightarrow (1). Suppose that T is a (\star, \star') -linked overring of D . Let $n \geq 1$ be an integer and $Q \in \text{QSpec}^{\star'[n]}(T[n])$. Set $Q_0 = Q \cap T$ which is a quasi- $\tilde{\star}'$ -prime ideal of T (or equal to zero). Then $P := Q_0 \cap D$ is a quasi- $\tilde{\star}$ -prime ideal of D by the observation before the theorem (or equal to zero). Thus D_P is a quasi-Prüfer domain by [7, Lemma 2.1]. Since $D_P \subseteq T_{Q_0}$, we find that T_{Q_0} is a universally catenarian domain by [2]. Thus $T[n]_Q = T_{Q_0}[n]_{QT_{Q_0}[n]}$ is a catenarian domain. Consequently $T[n]$ is $\star'[n]$ -catenary for each integer $n \geq 1$, that is T is an $\tilde{\star}'$ -universally catenarian domain. \square

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